Lecture 15

Tuesday, February 8, 2022 8:55 PM

* Prayer
* Spiritual thought

Partial derivatives : $f(x,y,z)$, f_x , f_y , f_z , f_{xx} , ...

Ex $f(x,y,z) = x^2 + 2yz + z^2x$

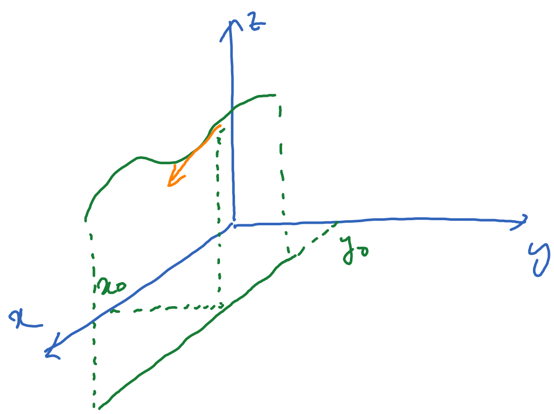
$f_x(1,2,3) = ?$

$f_x(x,y,z) = 2x + z^2$

$f_x(1,2,3) = 2(1) + 3^2 = 11$

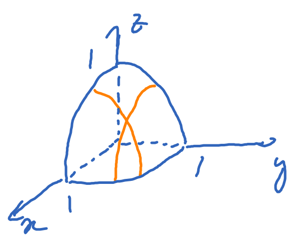
* Geometric meaning of partial derivatives :

$$f_x(x_0,y_0) = \frac{d}{dx} f(x,y_0)\Big|_{x=x_0} = \lim_{h\to 0} \frac{f(x_0+h,y_0) - f(x_0,y_0)}{h}$$



= slope of the curve above the line $y = y_0$ on the graph.

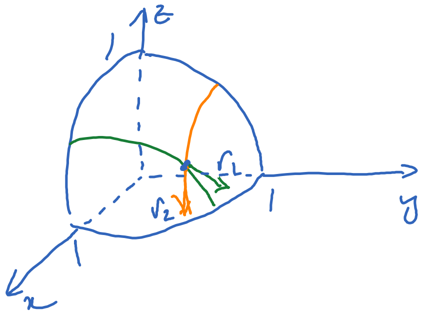
Ex



$f(x,y) = \sqrt{1-x^2-y^2}$, $x \geqslant 0, y \geqslant 0$

$f_x, f_y < 0$

Tangent plane



What is the equation of the plane tangent to the graph of $f(x,y)$ at the point $(x_0, y_0, z_0 = f(x_0,y_0))$?

The equation of the green curve is

$$r(t): \begin{cases} x = x_0 \\ y = t \\ z = f(x_0, t) \end{cases}$$

Tangent vector $= r'(t) = \langle 0, 1, f_y(x_0, t) \rangle$

At (x_0, y_0, z_0): $t = y_0 \rightsquigarrow$ tangent vector $r_1 = \langle 0, 1, f_y(x_0, y_0) \rangle$

Similarly, $r_2 = \langle 1, 0, f_x(x_0, y_0) \rangle$

Normal vector of the plane $= r_1 \times r_2 = \langle f_x, f_y, -1 \rangle$

Equation of tangent plane:

$$f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0) + (-1)(z - z_0) = 0$$

$$\rightsquigarrow \boxed{z = z_0 + f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0)}$$

Ex: $x^2+2y^2+z^2=1$ $(*)$

Tangent plane to this ellipsoid at $\left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right)$?

$z = z(x,y)$

Need $z_x\left(\frac{1}{2}, -\frac{1}{2}\right)$, $z_y\left(\frac{1}{2}, -\frac{1}{2}\right)$.

Differentiate $(*)$ wrt x: $2x + 0 + 2zz_x = 0$

$$\rightsquigarrow z_x = -\frac{x}{z} \rightsquigarrow z_x\left(\frac{1}{2}, -\frac{1}{2}\right) = -\frac{1/2}{1/2} = -1$$

Differentiate $(*)$ wrt y: $0 + 4y + 2zz_y = 0$

$$\rightsquigarrow z_y = -\frac{2y}{z} \rightsquigarrow z_y\left(\frac{1}{2}, -\frac{1}{2}\right) = -\frac{2(-1/2)}{1/2} = 2$$

Eq:

$$z = z_0 + z_x(x-x_0) + z_y(y-y_0)$$

$$= \frac{1}{2} - \left(x - \frac{1}{2}\right) + 2\left(y + \frac{1}{2}\right)$$

$$\rightsquigarrow -x + 2y - z = -\frac{3}{2}$$

* Differential

$$y = f(x) \rightsquigarrow dy = f'(x)\,dx$$

$$z = f(x,y) \rightsquigarrow dz = \underbrace{f_x\,dx}_{\text{partial differential}} + \underbrace{f_y\,dy} \text{ : total differential}$$